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Higher Order Theory for Vibrations of Thick Plates

A. V. Krishna Murty*

Indian Institute of Science, Bangalore, India

THE thick plate is basically a three-dimensional problem. The advantages of being able to treat it as a two-dimensional problem attracted research workers in the past to develop two-dimensional models for the analysis of thick plates. In 1945, Reissner[1] brought out that secondary effects such as transverse shear deformation can assume considerable significance in the bending analysis of elastic plates. In 1951, Mindlin[2] gave a theory for vibration analysis of plates, including the effects of transverse shear and rotary inertia. This formulation involved an unknown coefficient, which was determined by comparison with Lamb's exact solution for an infinite plate. Narasimha Murthy[3] developed an alternate formulation without involving any unknown constant. More recently, Srinivas[4] has carried out extensive studies of thickness effects on vibrations of plates. These studies have shown that, in certain ranges of the parameters involved, application of thin-plate theory can lead to significant errors in the prediction of the frequency spectrum. Thus, there is a need for a theoretical model, with flexibility for incorporating the required level of refinement and some guidelines for the choice of the level of refinement. In this Note, one such theory is presented. Starting from three-dimensional equations of motion, a hierarchy of sets of two-dimensional equations of motion representing the thick-plate behavior to different degrees of approximation have been derived. This derivation proceeds somewhat on similar lines followed by the author in Refs. 5-7 for the case of vibrations of short beams.

Formulation

In a homogeneous, linearly elastic thick plate the three displacement U, V, and W are, in general, functions of x, y, and z. It will be possible to construct a two-dimensional model, if the displacements U, V, and W can be expressed in terms of the midplane displacement u, v, and w and other generalized displacements which depend on x and y. This can be done by adopting the general approach of Ref. 5 with one difference. As in Ref. 5, if the transverse deflection is considered as a sum of two components w_b and w_s, one part w_b corresponding to the classical bending and the second part w_s corresponding to a shear strain which is constant across the thickness, then the second part w_s gives rise to nonzero complementary shear stresses at the top and bottom surfaces of the plate. This results in violation of the stress-free surface conditions and therefore the second component w_s cannot be included. U, V, W, are taken in terms of the midplane displacements u, v, w, as

$$U=u-z\frac{\partial w}{\partial x}+\sum_{n=1,2,3,\ldots} p_n\emptyset_n \tag{1a}$$

$$V=v-z\frac{\partial w}{\partial y}+\sum_{n=1,2,3,\ldots} p_n\psi_n \tag{1b}$$

$$W=w \tag{1c}$$

The midplane displacements u, v, and w and $\emptyset_n$ and ψ_n functions depend upon x and y only. Second terms in the expressions for U and V, correspond to the classical bending of thin plates. $\emptyset_n$, ψ_n terms are included in order to provide for arbitrary variation of U and V across the thickness. Thickness-wise variation of W is ignored as its effect is small on transverse natural frequencies.

As the primary interest here is in the bending vibrations of thick plates, U and V are considered to be antisymmetric in z. Hence $p_n=0$ at $z=0$. At the free surfaces $z=\pm t/2$, the shear stress is zero, $\mathrm{d}p_n/\mathrm{d}z=0$. To satisfy these conditions p_n are chosen as

$$p_n=\xi^{2n+1}-\frac{2n+1}{2^{2n}}\xi,\quad n=1,2,3,\ldots,\infty \tag{2}$$

where $\xi=z/t$.

Introducing Eqs. (1) and (2) in the standard three-dimensional stress-strain and strain-displacement relationships, and using the Hamilton principle, the governing equations for transverse vibrations may be deduced as

$$\begin{aligned}\lambda I\left[\frac{\partial^4 w}{\partial x^4}+2(\mu+2g)\frac{\partial^4 w}{\partial y^2\partial x^2}+\frac{\partial^4 w}{\partial y^4}\right]+\rho A\frac{\partial^2 w}{\partial t^{\prime}}\\ -\rho I\left[\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right]+\sum_{n=1,2,3,\ldots}^{M}\left[\lambda P_n\left\{\frac{\partial^3\emptyset_n}{\partial x^3}\right.\right.\\ \left.+\frac{\partial^3\psi_n}{\partial y^3}+(\mu+2g)\left(\frac{\partial^3\emptyset_n}{\partial x\partial y^2}+\frac{\partial^3\psi_n}{\partial y\partial x^2}\right)\right\}\\ \left.-\rho P_n\left(\frac{\partial^3\psi_n}{\partial y\partial t^2}+\frac{\partial^3\emptyset_n}{\partial x\partial t^2}\right)\right]=R(x,y,t)\end{aligned} \tag{3}$$

$$\begin{aligned}\lambda P_m\left[\frac{\partial^3 w}{\partial x^3}+(\mu+2g)\frac{\partial^3 w}{\partial x\partial y^2}\right]-\rho P_m\frac{\partial^3 w}{\partial x\partial t^2}\\ +\sum_{n=1,2,\ldots}^{M}\left\{\lambda E_{mn}\left[\frac{\partial^2\emptyset_n}{\partial x^2}+g\frac{\partial^2\emptyset_n}{\partial y^2}+(\mu+g)\frac{\partial^2\psi_n}{\partial x\partial y}\right]\right.\\ \left.-\lambda g S_{mn}\emptyset_n-\rho E_{mn}\frac{\partial^2\emptyset_n}{\partial t^2}\right\}=0,\ m=1,2,3,\ldots,M\end{aligned} \tag{4}$$

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*Associate Professor, Dept. of Aeronautical Engineering.

$$\lambda P_m \left[\frac{\partial^3 w}{\partial y^3} + (\mu+2g)\frac{\partial^3 w}{\partial y \partial x^2}\right] - \rho P_m \frac{\partial^3 w}{\partial y \partial t^2} + \sum_{n=1,2,..}^{M} \left\{\lambda E_{mn}\left[\frac{\partial^2 \psi_n}{\partial y^2} + g\frac{\partial^2 \psi_n}{\partial x^2} + (\mu+g)\frac{\partial^2 \emptyset_n}{\partial x \partial y}\right] - \lambda g S_{mn}\psi_n - \rho E_{mn}\frac{\partial^2 \psi_n}{\partial t^2}\right\} = 0, \quad m=1,2,3,...,M \tag{5}$$

where

$$\lambda = \frac{E}{1-\mu^2}, \qquad g = \frac{G}{\lambda}$$

$$I = \frac{h^3}{12}, \qquad A = h, \qquad P_n = -\int_{-h/2}^{h/2} z p_n \mathrm{d}z$$

$$E_{mn} = \int_{-h/2}^{h/2} p_m p_n \mathrm{d}z, \qquad S_{mn} = \int_{-h/2}^{h/2} \frac{\mathrm{d}p_n}{\mathrm{d}z}\frac{\mathrm{d}p_m}{\mathrm{d}z}\mathrm{d}z$$

and E denotes Young's modulus, μ Poisson's ratio, G shear modulus, and ρ the mass density. Two more equations for midplane vibrations, not coupled with transverse vibrations, are not reported here.

These equations govern the transverse vibrational behavior of the thick plate to different degrees of approximation depending upon the number of $\emptyset_n$ and ψ_n terms retained in the expansion for U and V. The governing equations, when M terms are retained, will be referred to as Mth order approximation equations or simply Mth-order approximation. It may be easily verified that when no $\emptyset_n$ or ψ_n terms are retained in the expansion for U and V, i.e., zeroth-order approximation, the governing equations reduce to the classical thin-plate theory. The first-order approximation corresponds to the thin-plate theory including the effects of transverse shear.[2]

Transverse Vibrations of a Rectangular Plate

The displacement field in a rectangular simply supported thick plate can be chosen as

$$w = w_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin(\omega t) \tag{6a}$$

$$\emptyset_n = \Phi_n \cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin(\omega t) \quad n=1,2,...,M \tag{6b}$$

$$\psi_n = \Psi_n \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin(\omega t) \quad n=1,2,...,M \tag{6c}$$

where a and b are half-wavelengths along the x and y directions, respectively, and are given by

$$a = L_1/m \qquad b = L_2/n$$

m and n being the number of half-waves along x and y directions. It may be easily verified that Eqs. (6) satisfy the simply supported condition (zero transverse deflection and zero in-plane normal stress conditions) at each edge. Substituting Eqs. (6) in Eqs. (3-5), and setting $R(x,y,t) = 0$, we get a set of homogeneous algebraic equations in the form

$$[K^{(M)}]\{\rho^{(M)}\} - \beta[M^{(M)}]\{\rho^{(M)}\} = 0 \tag{7}$$

where

$$\{\rho^{(M)}\}^T = [w_0, \emptyset_1, \psi_1, \emptyset_2, ..., \emptyset_M, \psi_M] \tag{8}$$

Table 1 Frequency spectrum to various orders of approximation

a/b	a/t	b/t	Thin-plate theory, $M=0$	First order, $M=1$	Second order, $M=2$	Third order, $M=3$
1	50	50	0.01298	0.01152	0.01152	0.01152
				8654.0	8642.8	8642.8
				8596.5	8944.2	8944.2
					81025	77783
					81075	77828
						250060
						250080
0.25	20	80	0.02286	0.01947	0.01947	0.01947
				1387.2	1385.4	1385.4
				1455.8	1453.8	1453.8
					12967	12448
					12981	12461
						40010
						40019

$$\beta = \omega^2 \rho a^2 / \lambda \tag{9}$$

Table 1 gives the frequency parameters, obtained from Eqs. (7) to various orders of approximation, for two typical cases of rectangular plates. Higher-order approximations $(M>1)$ bring out more frequencies than are accounted for by the classical thin-plate theory. Recalling that half-wavelengths a and b are made use of in the choice of the displacement functions [see Eq. (6)] and recognizing the possibility of having any integral number of half-waves along each side, we note that each frequency parameter in Table 1 corresponds to an infinite set of frequencies of a particular type of mode shape. The classical thin-plate theory $(M=0)$ predicts only one set, whereas higher-order approximation brings out more sets. In general, Mth order approximation equations brings out $(2M+1)$ sets of frequencies. The mode shape corresponding to the lowest frequency parameter by any order approximation involves primarily transverse motion. The other sets of frequencies are generally very high and involve primarily in-plane motions coupled with small transverse motion.

Also, Table 1 presents the convergence trend. The fundamental frequency is estimated correct to four significant figures by the first-order approximation itself. A similar convergence trend is observed in the range of the parameters $0.1 < a/b < 2$ and $20 < a/t < 500$; thus confirming the adequacy of the first-order theories [1,2] for natural vibration analysis of isotropic thick plates.

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